Preference Relations and Choice Rules

Econ 3030 Fall 2025

Lecture 1

Outline

1. Logistics
2. Introduction to Consumer Theory
3. Binary Relations
4. Preferences
5. Choice Correspondences
6. Revealed Preferences

Logistics

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Class Time and Location
Tuesday and Thursday, 9:00am-10:15am
location: Posvar 4940
Recitation Times
Thursday 1:00pm-2:15pm, Posvar 4515

Class Webpage `http://www.pitt.edu/~luca/ECON3030/`
(Canvas for annotated lectures)

Logistics

Textbooks:
Kreps, Microeconomic Foundations I: Choice and Competitive Markets
Mas-Collel, Whinston and Green: Microeconomic Theory.
Another useful book: Rubinstein, Lecture Notes in Micro Theory.
Personal favorite: Gerard Debreu: Theory of Value.

Grading

- *Problem Sets: 10%*
  Weekly, due on Canvas by the beginning of each Tuesday classs.
  DO NOT *look for answers... These are for you to learn what you can and cannot do; we can help you get better only if we know your limits.*
- *Midterm: 30%*
- *Final: 60%*

- Working in groups is strongly encouraged, but turn in problem sets individually (start working on exercises on your own, and then get together to discuss).

Goals of the Micro Theory Sequence

- Learn and understand the microeconomic theory every academic economist needs.
- Stimulate interest in micro theory as a field.
- Enable you to read papers that use theory, and go to theory research seminars.
- ECON3030 covers the following standard topics:
  - consumers: preferences, choices, utility function representation, utility maximization, demand theory, aggregation, decision making under uncertainty;
  - firms: production, profit maximization, aggregation;
  - general equilibrium theory: Pareto efficiency, competitive markets, Walrasian equilibrium, First and Second Welfare Theorems, existence of equilibrium, uniqueness of equilibrium, uncertainty and time, Arrow-Debreu economy.
- In the Spring, Beixi Zhou will cover game theory and information economics.
- Questions?

A One Slide Introduction to Consumer Theory

Imagine rational consumers who can choose objects from some abstract set.

Economists describe them in three different ways:

1. We describe the consumers' taste, their preferences, and those determine their choices.
2. We say consumers maximizes a utility function over their possible choices.
3. We describe a choice functional that tells us how each consumer behaves.

- The first few lectures are about showing under what conditions these ways of modeling rational consumers are equivalent.

First, we describe preferences and choices formally, and start talking about possible connections between them.

Binary Relations

- Mathematically, preferences are a special case of a binary relation.
- A binary relation on some set is a collection of ordered pairs of elements of that set.

Definition

$R \subseteq X \times Y$ is a binary relation from X to Y.
We write "xRy" if $(x, y) \in R$ and "not xRy" if $(x, y) \notin R$.
When $X = Y$ and $R \subseteq X \times X$, we say R is a binary relation on X.

Exercise

Suppose R, Q are two binary relations on X. Prove that, given our notation, the following are equivalent:

1. $R \subseteq Q$
2. For all $x, y \in X$, $xRy \Rightarrow xQy$.

NOTE

Exercises are mainly meant as practice for you, distinct from graded problem sets.

Examples of Binary Relations

A Function Is a Binary Relation

Suppose $f : X \to Y$ is a function from X to Y.

- Then the binary relation $R \subseteq X \times Y$ defined by
$$xRy \Leftrightarrow f(x) = y$$
is the graph of f.
- One can think of a function as a binary relation R from X to Y such that for each $x \in X$, there exists exactly one $y \in Y$ such that $(x, y) \in R$.

Examples of Binary Relations

Weakly greater than

Suppose $X = \{1, 2, 3\}$

- Consider the binary relation

$$R \subseteq \{1, 2, 3\} \times \{1, 2, 3\}$$

  defined as follows

$$R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}.$$

- R is the binary relation "is weakly greater than," or $\geq$.
- We can represent it graphically as:

3			●
2		●	●
1	●	●	●
$\geq$	1	2	3

Examples of Binary Relations

Equal to and Strictly less than

Suppose $X = \{1, 2, 3\}$.

- The following graphically represents the binary relation $=$, or “is equal to”

3			$\bullet$
2		$\bullet$	
1	$\bullet$		
$=$	1	2	3

- The following would represent the binary relation $<$ or “is strictly less than”

3	$\bullet$	$\bullet$	
2	$\bullet$		
1			
$<$	1	2	3

Dual, Asymmetric, and Symmetric Components

Definitions

Given a binary relation R on X.

- The dual of R, denoted R', is defined by $xR'y$ if and only if yRx.
- The asymmetric component of R, denoted P, is defined by xPy if and only if xRy and not yRx.
- The symmetric component of R, denoted I, is defined by xIy if and only if xRy and yRx.

Example

Suppose $X = \mathbf{R}$ and R is the binary relation $\geq$, or "weakly greater than"

- The dual is $\leq$ or "weakly less than," because $x \geq y$ if and only if $y \leq x$.
- The asymmetric component is $>$ or "strictly greater than" because $x > y$ if and only if $x \geq y$ and not $y \geq x$. (Verify this).
- The symmetric component is $=$ or "equal to" because $x = y$ if and only if $x \geq y$ and $y \geq x$.

Properties of Binary Relations

Definitions

A binary relation R on X is:

- complete if, for all $x, y \in X$, xRy or yRx;
- reflexive if, for all $x \in X$, xRx;
- irreflexive if, for all $x \in X$, not xRx;
- symmetric if, for all $x, y \in X$, xRy implies yRx;
- asymmetric if, for all $x, y \in X$, xRy implies not yRx;
- antisymmetric if, for all $x, y \in X$, xRy and yRx imply $x = y$;
- transitive if, for all $x, y, z \in X$, xRy and yRz imply xRz;
- negatively transitive if, for all $x, y, z \in X$, not xRy and not yRz imply notxRz;
- quasi-transitive if, for all $x, y, z \in X$, xPy and yPz imply xPz;
- acyclic if, for all $x_1, x_2, \ldots, x_n \in X$, $x_1 P x_2$, $x_2 P x_3$, $\ldots$, and $x_{n-1} P x_n$ imply $x_1 R x_n$.

Properties of Binary Relations: Exercises

Exercise

Suppose $X = \mathbb{R}$.

- Show that the binary relation $\geq$ is reflexive, complete, antisymmetric, transitive, and negatively transitive; $\geq$ is not asymmetric.
- Show that the binary relation $>$ is irreflexive, asymmetric, antisymmetric, transitive, negatively transitive, quasi-transitive, and acyclic; $>$ is not reflexive, not complete, and not symmetric.

Exercise

Suppose R is complete. Prove the following: If R is transitive, then R is quasi-transitive. If R is quasi-transitive, then R is acyclic.

Exercise

Let X be the set of all living people. Are the following relations on X reflexive? symmetric? transitive? complete?

"is married to" (assuming monogamy)	"is the son or daughter of"
"is an ancestor or descendant of"	"is taller than"

Orders: Definitions

Definition

A binary relation on X is a preorder if it is reflexive and transitive.

Definition

A binary relation on X is a weak order if it is complete and transitive.

Definition

A binary relation on X is a linear order if it is complete, transitive, and antisymmetric.

Orders: Examples

Exercise

Define the binary relation $\geq$ on $\mathbb{R}^2$ by: $x \geq y \Leftrightarrow x_1 \geq y_1$ and $x_2 \geq y_2$.
Verify that $\geq$ is a preorder on $\mathbb{R}^2$. Verify that $\geq$ is not a weak order on $\mathbb{R}^2$.

Exercise

Define $\geq^\dagger$ on $\mathbb{R}^2$ by: $x \geq^\dagger y \Leftrightarrow x_1 > y_1$ or, $x_1 = y_1$ and $x_2 \geq y_2$
Verify that $\geq^\dagger$ is a linear order on $\mathbb{R}^2$ (this is sometimes called lexicographic ordering).

Upper and Lower Contour Sets

Definitions

Given a binary relation R on X, the upper contour set of $x \in X$ is

$$\{y \in X : yRx\}.$$

Given a binary relation R on X, the lower contour set of $x \in X$ is

$$\{y \in X : xRy\}.$$

Exercise

Suppose R is a preorder on X (reflexive and transitive). Prove that if xRy, then the lower contour set of y is a subset of the lower contour set of x; that is, prove that

$$xRy \Rightarrow \{z \in X : yRz\} \subseteq \{z \in X : xRz\}.$$

Preferences As Binary Relations

- Assume there is some abstract set X of objects consumers consider.
- The consumers' taste, their preferences, describe how they rank any two elements in X.
- In other words, preferences are described mathematically by a binary relation $\succsim$ on X.
- $x \succsim y$ reads: the decision maker (DM) thinks x is at least as good as y (weakly prefers x to y).
- Kreps: "A preference relation expresses the consumer's feelings between pair of objects in X".

Remark

For any $x, y \in X$, the consumer is willing to say which of the following holds:

1. $x \succsim y$ but not $y \succsim x$;
2. $y \succsim x$ but not $x \succsim y$;
3. $x \succsim y$ and $y \succsim x$;
4. neither $x \succsim y$ nor $y \succsim x$.

Preference Relation

Definition

A binary relation $\succsim$ on X is a preference relation if it is a weak order, i.e. *complete* and *transitive*.

- Transitivity can be thought of as implied by rationality in the sense of some coherence of taste.
- Completeness, on the other hand, is harder to justify as a consequence of rationality: what is wrong with not being able to rank all possible pairs?
  - it is very powerful, as it rules out inability to rank (4. in the previous remark is not possible).
  - Because of this, all other assumptions have a global impact.

Question 1, Problem Set 1: Transitivity follows from weaker properties.

Prove the following: if $\succsim$ is asymmetric and negatively transitive, then $\succsim$ is transitive.

NOTE

Problem Set 1 is posted on Canvas.

Upper and Lower Contour Sets

Definition

The upper contour set of x (denoted $\succsim(x)$) consists of the elements of X that are weakly preferred to x according to $\succsim$:

$$\succsim(x) = \{y \in X : y \succsim x\}$$

- These are all consumption bundles that are at least as good as x.

Definition

The lower contour set of x (denoted $\precsim(x)$) consists of the elements of X that x is weakly preferred to according to $\succsim$:

$$\precsim(x) = \{y \in X : x \succsim y\}$$

- These are all consumption bundles that x is at least as good as.

Preference Relations

Definitions

For any preference relation $\succsim$ on X,

- $\precsim$ denotes the dual of $\succsim$, defined by
$$x \precsim y \Leftrightarrow y \succsim x;$$
- $\succ$ denote the asymmetric component of $\succsim$, defined by
$$x \succ y \Leftrightarrow [x \succsim y \text{ and not } y \succsim x];$$
- $\sim$ denote the symmetric component of $\succsim$, defined by
$$x \sim y \Leftrightarrow [x \succsim y \text{ and } y \succsim x].$$

NOTE

$\precsim$, $\succ$, and $\sim$ are all defined starting from $\succsim$.

- $x \succ y$ reads: DM *strictly prefers* x to y;
- $x \sim y$ reads: DM *is indifferent* between x and y.

Preference Relations: Exercises

Exercise

Let $X = \{a, b, c\}$. Are the following binary relations complete and/or transitive?

1. $\succsim = X \times X$;
2. $\succsim = \emptyset$;
3. $\succsim = \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (b, c)\}$;
4. $\succsim = \{(a, a), (b, b), (c, c), (a, b), (b, c), (c, a)\}$;
5. $\succsim = \{(a, b), (b, c), (a, c)\}$.

Question 2, Problem Set 1.

Prove that if $\succsim$ is a preference relation (i.e. it is complete and transitive), then:

1. $\precsim$ is a preference relation;
2. $\succ$ is asymmetric and transitive;
3. $x \succsim y$ and $y \sim z$ imply $x \succsim z$;
4. $x \succsim y$ and $y \succ z$ imply $x \succ z$.

Choices and Correspondences

- A preference relation describes DM's rankings of any two hypothetical pairs.
- Next, we describe how the decision maker could choose from a given set of alternatives.
  - Ideally, this would produce observable outcomes that are useful to test theories.
- Afterwards we will worry about the connection between preferences and choices.
- The mathematical object that describes choices is a correspondence.

Definition

A correspondence φ from X to Y is a mapping from X to 2^Y; that is, $\varphi(x) \subseteq Y$ for every $x \in X$.

Choice Rules

Definition

A choice rule for X is a correspondence

$$C : 2^X \setminus \{\emptyset\} \to X \quad \text{such that} \quad C(A) \subseteq A \text{ for all } A \subseteq X.$$

Interpretation

- C describes which items could be selected from a set of available objects.
- Susbsets of $2^X \setminus \{\emptyset\}$ are "menus" or "budgets".
  - A particular $A \in 2^X \setminus \{\emptyset\}$ represents the available options (i.e. affordable consumption).
- Given a budget, $C(A)$ is the set of options DM might choose from it.
  - If $C(A)$ has more than one element, she could choose any of them (but not all of them at once).

Remark

- Whether $C(A)$ is actually observable or not is unclear. At best, one observes the decision maker select an element of $C(A)$.
- So, one can include elements in the choice set from observation, but one cannot exclude them without making additional assumptions on $C(A)$.

Choice Rules: An Example

Example

Let $X = \{apple, banana, carrot, dessert, elephant\}$.

- If $C(\{a, b, c\}) = \{a, b\}$, DM could choose either the apple or the banana from a basket containing an apple, a banana, and a carrot;
- This is not interpreted as meaning the decision maker will consume both the apple and the banana.
- By definition, this means DM will consume only one between the apple and the banana, but we do not know which one.

From Preferences to Choices: Induced Choice Rules

Definition

Given a binary relation $\succsim$, the induced choice rule $C_{\succsim}$ is defined by

$$C_{\succsim}(A) = \{x \in A : x \succsim y \text{ for all } y \in A\}.$$

- This is a natural method to construct a choice rule from a binary relation: DM chooses something weakly preferred to the other available alternatives.
- This definition answers one of our questions: the induced choice rule gives a choice procedure that is consistent with a given preference relation.

Induced Choice Rules: Examples

Definition

Given a binary relation $\succsim$, the induced choice rule $C_{\succsim}$ is defined by

$$C_{\succsim}(A) = \{x \in A : x \succsim y \text{ for all } y \in A\}.$$

Example

Suppose $X = \{1, 2, 3, \ldots\}$ and consider $\geq$, $\leq$, and $>$. Then:

- $C_{\geq}(A) = \max A$ if A is finite and $C_{\geq} = \emptyset$ if A is infinite.
- $C_{\leq}(A) = \min A$ for all sets A.
- $C_{>}(A) = \emptyset$ for all A.

Example

Let $X = \{a, b, c\}$ and let $\succsim = \{(a, a), (b, b), (c, c), (a, b), (b, c), (c, a)\}$.
What is $C_{\succsim}(\{a, b, c\})$?

Observable Choices

Definition

Given a binary relation $\succsim$, the induced choice rule $C_{\succsim}$ is defined by

$$C_{\succsim}(A) = \{x \in A : x \succsim y \text{ for all } y \in A\}.$$

- This definition starts with a binary relation and derives a choice rule.
- A preference relation describes the decision maker's taste. It is not observable.
- Therefore, the induced choice rule is also unobservable: the only way an observer can know $C_{\succsim}$ is to know $\succsim$.

Non Empty Choice Rules

Definition

The choice rule C is non-empty if $C(A) \neq \emptyset$ for all non-empty $A \subseteq X$.

Question 3, Problem Set 1.

Prove that if $\succsim$ is a preference relation, then $C_{\succsim}(A) \neq \emptyset$ whenever A is finite.

- Given a preference relation, the corresponding induced choice rule is non-empty on finite menus.

Question 4, Problem Set 1.

Prove that $\succsim$ is complete and acyclic (but not necessarily transitive) if and only if $C_{\succsim}(A) \neq \emptyset$ whenever A is finite.

Deduce Preferences From Choice Rules

GOAL

Observe DM's behavior and use these observations to learn about their preferences.

- This is the opposite direction of the induced choice rule.
- One starts with a particular choice rule and then deduces whether or not it could have been induced by some preference relation.
  - This is similar to the idea of finding a theory which is consistent with available data.
- One wants to find some possible rationale behind particular choice patterns.
- Naturally, this is not possible for any arbitrary choice rule. Only some choice procedures are consistent with rationality.
- The next step is to define a choice rule that could come from some preference.

Rationalizable Choice Rules

Definition

A choice rule C is rationalized by $\succsim$ if $C = C_{\succsim}$ and $\succsim$ is a preference relation.

- This is a choice rule that behaves as if it maximizes some complete and transitive binary relation among the available alternatives.

Definition

A choice rule C is rationalizable if there exists a preference relation $\succsim$ such that $C = C_{\succsim}$.

- If a choice rule is not rationalizable, there is no preference behind it.
- If a choice rule is rationalizable, one may still not be able to identify the preferences behind it because there may be many possible $\succsim$ that are consistent with it.
- Next, define the preference relation consistent with some given choice rule.

Revealed Preferences

Definition

Given a choice rule C, its revealed preference relation $\succsim_C$ is defined by

$$x \succsim_C y \text{ if there exists some } A \text{ such that } x, y \in A \text{ and } x \in C(A).$$

- $x \succsim_C y$ reads "x is revealed preferred to y".
- IDEA: if DM chooses x when y is available one says that x is revealed to be weakly preferred to y.
- One builds $\succsim_C$ observing choices **and** the menus they come from.
  - These are the preferences consistent with particular choice behavior.

Remark

The definition says that there exists some menu including x and y where x is chosen, not that x is chosen in all menus including x and y.

- Revealed preferences do not necessarily reflect DM's levels of happiness or well-being; they only reflect what DM decided to do.
  - If we observe her order salad at a restaurant, we say she reveals to prefer salad to pasta, but that is not necessarily the same as saying she thinks the salad tastes better.

Revealed Preferences Are Rationalizable Preferences

Proposition

If C is rationalized by (a complete and transitive) $\succsim$, then $\succsim = \succsim_C$.

Proof.

Let $\succsim$ be a preference relation which rationalizes C; that is, for all $A \subseteq X$:

$$C(A) = C_{\succsim}(A) = \{x \in A : x \succsim y \text{ for all } y \in A\}$$

- Suppose $x \succsim y$. We need to show $x \succsim_C y$.
  - Since $\succsim$ is complete and transitive, we have $x \succsim x$.
  - Thus $x \succsim z$ for all $z \in \{x, y\}$, so $x \in C_{\succsim}(\{x, y\})$.
  - Since $C = C_{\succsim}$, this implies $x \in C(\{x, y\})$.
  - Therefore x is revealed preferred to y: $x \succsim_C y$.
- Suppose $x \succsim_C y$. We need to show $x \succsim y$.
  - By definition, there exists some set A with $x, y \in A$ and $x \in C(A)$.
  - Then $x \in C_{\succsim}(A)$ because $C = C_{\succsim}$.
  - By definition, $x \succsim z$ for all $z \in A$.
  - But $y \in A$, so $x \succsim y$.

$\square$

Revealed Preferences Are Rationalizable Preferences

Proposition

If C is rationalized by (a complete and transitive) $\succsim$, then $\succsim = \succsim_C$.

- Now the proof is done, what does this mean?

Remark

The only preference relations that can rationalize C are revealed preference relations.

- To check whether or not a choice rule is rational, all one needs to do is check whether or not it acts as if it were "maximizing" its revealed preference relation.
- Contrapositively, if $\succsim_C$ does not rationalize C, then no other preference relation will rationalize C.
- This imposes restrictions on DM's behavior: from some observed choices we can deduce what other (unobserved) choices will have to be.

Example

Let $X = \{a, b, c\}$ and let C be a rationalizable choice rule such that

$$C(\{a\}) = \{a\},\ C(\{b\}) = \{b\},\ C(\{c\}) = \{c\}, \text{ and } \begin{array}{c} C(\{a, b\}) = \{a\} \\ C(\{b, c\}) = \{b\} \\ C(\{a, c\}) = \{a\} \end{array}.$$

Can one predict what $C(\{a, b, c\})$ looks like with the help of the previous result?

Claim: $C(\{a, b, c\}) = \{a\}$.

Exercise

Prove the claim

Next Class

- Weak Axiom of Revealed Preference and Rationalizable Choice Rules
- Introduction to Utility Functions